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OPTIMAL LINEAR SWITCHING FOR SINGULAR LINEAR SYSTEMS

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ABSTRACT

The bounded-input control of a linear, time-invariant system with quadratic performance index is discussed from the standpoint of simple implementation. In particular the admissible control is constrained to be a linear function of the states fed back through a saturating amplifier. The resulting controlled system not only exhibits the usual maximum-effort made of operation, but also the terminal singular behavior where linear control is in evidence.

INTRODUCTION

I-1 Background

The optimal linear switching problem has been studied in some detail as an approximation to the true optimal control--sub-optimal control--for systems with quadratic performance indices [1,2,3,4]. The optimal relay control of Jen-Wei [1], although theoretically incorrect[1], exhibited the so-called singular solutions (slippage modes) anticipating the later interest in depth in singular solutions [5,6]. These previous efforts lead naturally to the question: If a simple control scheme (viz., linear switching) is highly desirable, why not restrict the class of admissible controls to those satisfying this control scheme at the outset? The method for obtaining the terminal singular solution in the single-input case is at least conceptionally simple and computationally possible [5]; however, outside of the region of admissible linear control, the switching curve is difficult to obtain--and the switching is difficult to implement as well.

The proposed solution here might be called "sub-optimal" in that it is not the "best" that can be done when one has available an infinitude of resources. On the other hand it is truly optimal in the sense that the means of control is restricted at the outset with an eye to the eventual implementation (an implementation which is usually considered the practical compromise, anyway), and under this restriction the control is optimal. A distinct advantage of the linear switching obtained is that the control in no way depends on the system's initial conditions while still exhibiting the optimal singular behavior.

I-2 Formulation of the Problem

We consider a linear, time-invariant, single-input system described by the state equations

$$\dot{\underline{x}} = A\underline{x} + \underline{b}u \tag{1a}$$

and the input constraint

$$|u| \leq 1. \tag{1b}$$

Moreover, the control u is to be given by the function

$$u = f(v) \tag{2}$$

where

$$f(v) = v \quad \text{for} \quad |v| \leq 1, \tag{3a}$$

$$f(v) = +1 \quad \text{for} \quad v > 1, \tag{3b}$$

$$f(v) = -1 \quad \text{for} \quad v < -1, \tag{3c}$$

and the function v(t) is constrained to be a linear combination of the state variables:

$$v(t) = \underline{c}^T\underline{x}(t) = \sum_{i=1}^{n} c_i \, x_i(t). \tag{4}$$

The quadratic performance index is the functional[2]

$$J(\underline{c}) = \int_0^\infty \underline{x}^T \, Q \, \underline{x} \, dt, \tag{5}$$

and the problem is to take the system from any initial condition to the origin while (5) is minimized. The problem can be restated as that of finding the vector $\underline{c}$ in E^n (Euclidean n-space) which minimizes the functional (5) along the trajectories (1) and (2). Moreover, no magnitude constraints need be imposed on $\underline{c}$; it may be any vector in E^n.

II. THE OPTIMAL CONTROL

The problem as formulated above is quite easily handled by means of the Maximum Principle of Pontryagin [7]. In order to apply the Maximum Principle, we first form the Hamiltonian:

$$\mathcal{H}(\underline{\Psi}, \underline{x}, \underline{c}) \equiv \underline{\Psi}^T A\underline{x} + \underline{\Psi}^T \underline{b}\, f(\underline{c}^T \underline{x}) - \underline{x}^T Q\, \underline{x}, \qquad (6)$$

where the auxiliary vector $\underline{\Psi}$ (the conjugate momentum) is the solution of

$$\dot{\underline{\Psi}} = - \nabla_x \mathcal{H}\,. \qquad (7)$$

Equation (7) in view of (6) becomes

$$\underline{\Psi} = -A^T \underline{\Psi} + 2\, Q\underline{x} - \underline{\Psi}^T \underline{b}\, \nabla_x f(\underline{c}^T \underline{x}), \qquad (8a)$$

where the i^{th} component of $\nabla_x f(\underline{c}^T \underline{x})$ is given by

$$[\nabla_x f(\underline{c}^T \underline{x})]_i = \frac{\partial f(\underline{c}^T \underline{x})}{\partial x_i}\ ; \ i = 1, 2, \ldots, n. \qquad (8b)$$

The Maximum Principle states that the vector $\underline{c}$ which minimizes the functional (5) must necessarily maximize the Hamiltonian (6); hence, at the maximum

$$\frac{\partial \mathcal{H}}{\partial c_i} = 0, \ i = 1, 2, \ldots, n. \qquad (9)$$

In view of (6) conditions (9) become

$$\frac{\partial}{\partial c_i} [\underline{\Psi}^T \underline{b}\, f(\underline{c}^T \underline{x})] = 0, \ i = 1, 2, \ldots, n, \qquad (10a)$$

or

$$\underline{\Psi}^T \underline{b}\, \frac{\partial f(\underline{c}^T \underline{x})}{\partial c_i} = 0, \ i = 1, 2, \ldots, n. \qquad (10b)$$

The derivatives indicated in (8b) and (10b), because of (3) and (4), can be rewritten

$$\frac{\partial}{\partial x_i} f(\underline{c}^T \underline{x}) = \frac{\partial f(v)}{\partial v} \frac{\partial v}{\partial x_i} \tag{11a}$$

and

$$\frac{\partial}{\partial c_i} f(\underline{c}^T \underline{x}) = \frac{\partial f(v)}{\partial v} \frac{\partial v}{\partial c_i} \quad , \tag{11b}$$

respectively. Moreover, from (3)

$$\frac{\partial f(v)}{\partial v} = 1 \quad \text{for} \quad |v| \leq 1 \tag{12a}$$

and

$$\frac{\partial f(v)}{\partial v} = 0 \quad \text{for} \quad |v| > 1; \tag{12b}$$

furthermore, from (4)

$$\frac{\partial v}{\partial x_i} = c_i \tag{13a}$$

and

$$\frac{\partial v}{\partial c_i} = x_i \quad . \tag{13b}$$

Consequently, conditions (10) become

$$\underline{b}^T \underline{\Psi}\, x_i = 0 \quad \text{for } |v| \leq 1 \tag{14a}$$

and

$$0 = 0 \quad \text{for } |v| > 1; \tag{14b}$$

The optimal control can be obtained from any $\underline{c}$ such that for each state $\underline{x}$ in the state space

$$|\underline{c}^T \underline{x}| > 1, \tag{15}$$

which follows from (14b)--unless

$$\underline{b}^T \underline{\Psi} = 0, \tag{16}$$

as in (14a). Condition (16) is that of the singular solution; a control u = f(v) which realizes (16) <u>over a finite time interval</u> is a singular control.

In Appendix B is a derivation for the singular surface on which (16) holds:

$$\sum_{i=1}^{n} h_i \, x_i \, (t) = 0; \tag{17}$$

this linear equation describes an (n-1)-dimensional hyperplane in the n-dimensional state space. Obviously, the system may satisfy (17) if the initial conditions satisfy the condition

$$\underline{h}^T \, \underline{x}_o = 0, \tag{18}$$

where

$$h = \begin{bmatrix} h_1 \\ h_2 \\ . \\ . \\ . \\ h_n \end{bmatrix}, \; h_n = 1. \tag{19}$$

Moreover, the optimal singular control which maintains the system on the singular surface (when it is admissible) is given by

$$u = f(v), \tag{20a}$$

$$v = \underline{c}^T \, \underline{x}, \tag{20b}$$

and

$$\underline{c} = \underline{a} - \hat{\underline{h}} - k\,\underline{h}\ , \tag{20c}$$

where

$$\hat{\underline{h}} = \begin{bmatrix} 0 \\ h_1 \\ h_2 \\ \cdot \\ \cdot \\ \cdot \\ h_{n-1} \end{bmatrix}\ , \tag{21a}$$

$$\underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{bmatrix} \tag{21b}$$

is the negative of the transposed last row of the assumed companion form A-matrix[3] and

$$k = \frac{a_1 - c_1}{h_1}\ . \tag{21c}$$

This control with k arbitrary (i.e., with c_1 arbitrary) causes the system trajectory to remain on the singular hyperplane (17) whenever

$$|(\underline{a} - \hat{\underline{h}})^T\ \underline{x}| < 1 \tag{22}$$

and (18) is satisfied. If (22) is not satisfied, there is no control (among the admissible ones) which makes $\underline{b}^T\underline{\Psi} = 0$ over a finite time interval. Note, (20) determines all c_i's except c_1; moreover, the control signal on the singular hyperplane is independent of the value of c_1.[4] Letting

$$\underline{c}_s = \underline{a} - \underline{\hat{h}} \tag{23a}$$

and

$$\underline{c}_{ns} = -k\underline{h}, \tag{23b}$$

we have

$$\underline{c} = \underline{c}_s + \underline{c}_{ns}; \tag{24}$$

i.e., the control is a linear combination of singular and nonsingular controls. For all $\underline{x}$ on the admissible singular hyperplane, i.e., for $\underline{x}$ such that

$$\underline{h}^T \underline{x} = 0 \tag{25a}$$

and

$$|(\underline{a}-\underline{\hat{h}})^T \underline{x}| < 1, \tag{25b}$$

the optimal control is given by (23a):

$$\underline{u} = \underline{c}_s^T \underline{x} = (\underline{a}-\underline{\hat{h}})^T \underline{x}. \tag{26}$$

For $\underline{x}$ not satisfying (25) (<u>both</u> <u>25a</u> and <u>25b</u>),

$$\frac{\partial f(v)}{\partial v} = 0 \tag{27a}$$

or

$$|\underline{c}^T \underline{x}| > 1 \tag{27b}$$

must hold for (10) to be satisfied. Condition (27b) can be maintained (when appropriate) by taking[5]

$$c_1 = -\infty \ . \tag{28}$$

Of course, an infinite gain is difficult to obtain in practice, but from the simple implementation shown in Fig. 1, it is clear that a sufficiently large finite gain (+k) realizes the optimal control within any predetermined error. Moreover, limiting the gain (k) to that which is practically attainable assures that the optimal c_1 is chosen from a closed set.

The stability of the closed-loop system follows from the optimality of the control. The control given by (20c) is optimal; it thus provides a smaller value for the functional (5) than does the admissible control u = 0 (this is true for any $\underline{x}_o$). But if the original system (1) is stable (this property is assumed here), the integral (5) is bounded for u = 0; it is therefore bounded for the optimal control--a clear indication of stability. The only problem, then, is to assure the optimality of the control (20c); this assurance in general involves an investigation of the second variation--unless strong physical arguments can be mustered--which is often quite difficult. An alternative verification of stability could be undertaken by employment of the most recently obtained sufficient conditions for the Lur's Problem [8].

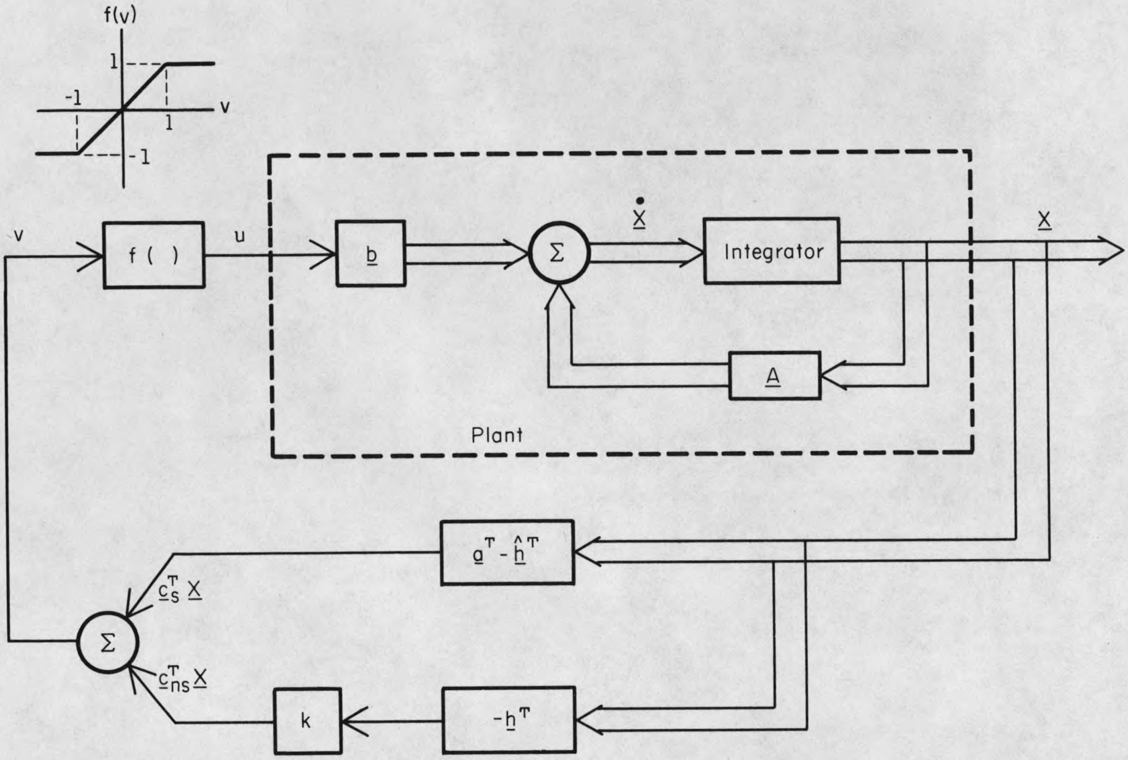

Fig. 1. Implementation of optimal linear switching.

III. ILLUSTRATIVE EXAMPLE

Consider the system characterized by

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -x_1 - 3x_2 + u$$

and

$$|u| \leq 1.$$

The performance criterion is to minimize the functional

$$J = \int_0^\infty (x_1^2 + x_2^2)\, dt,$$

while the system trajectory goes to the origin in a stable manner. The stable Euler equation is

$$x_1 + x_2 = 0,$$

and the feedback control which maintains it on the singular hyperplane (17) is

$$u = x_1 + 2x_2 - k(x_1+x_2).$$

The system is stable for $k > 0$ and optimal when $k \rightarrow \infty$; the implementation of this control is shown in Fig. 2. In Fig. 3, the phase plane behavior of the system trajectory is given for similar initial conditions ($x_{20} = 0$ and $x_{10} = 0.8, 1.2, 1.6$, and 2.0) and various values of gain ($k = 2, 20$, and 200). The linear region is indicated in each case as well as the shapes of the control curves (u vs. x_1) which provide the optimal. The pertinent feature which should be noticed is the extreme similarity between the curves obtained for $k = 20$ and $k = 200$, clearly indicating the sufficiency of a small, finite gain.

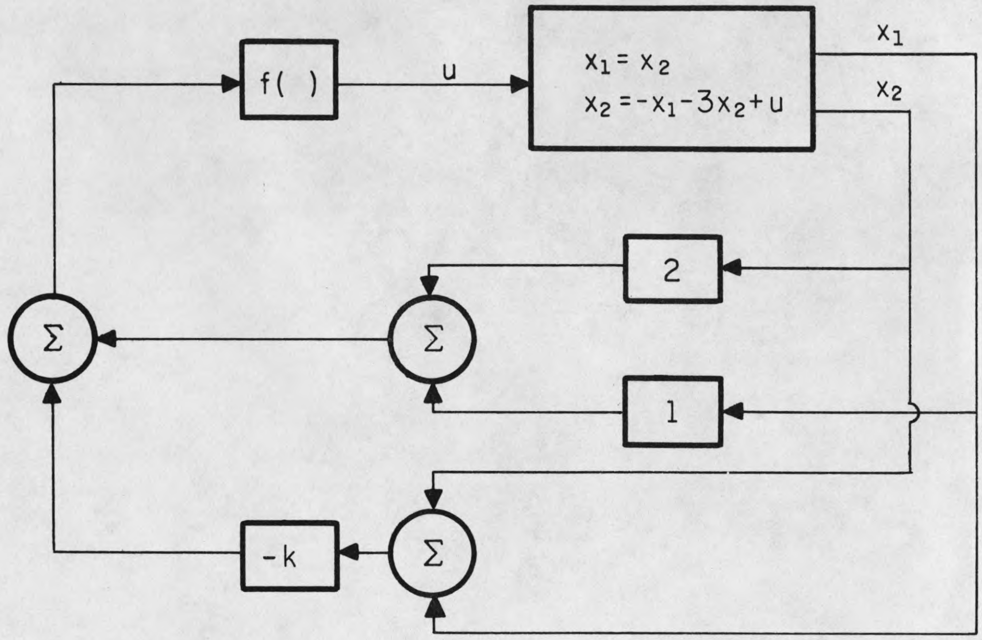

Fig. 2. Optimal control of second-order example.

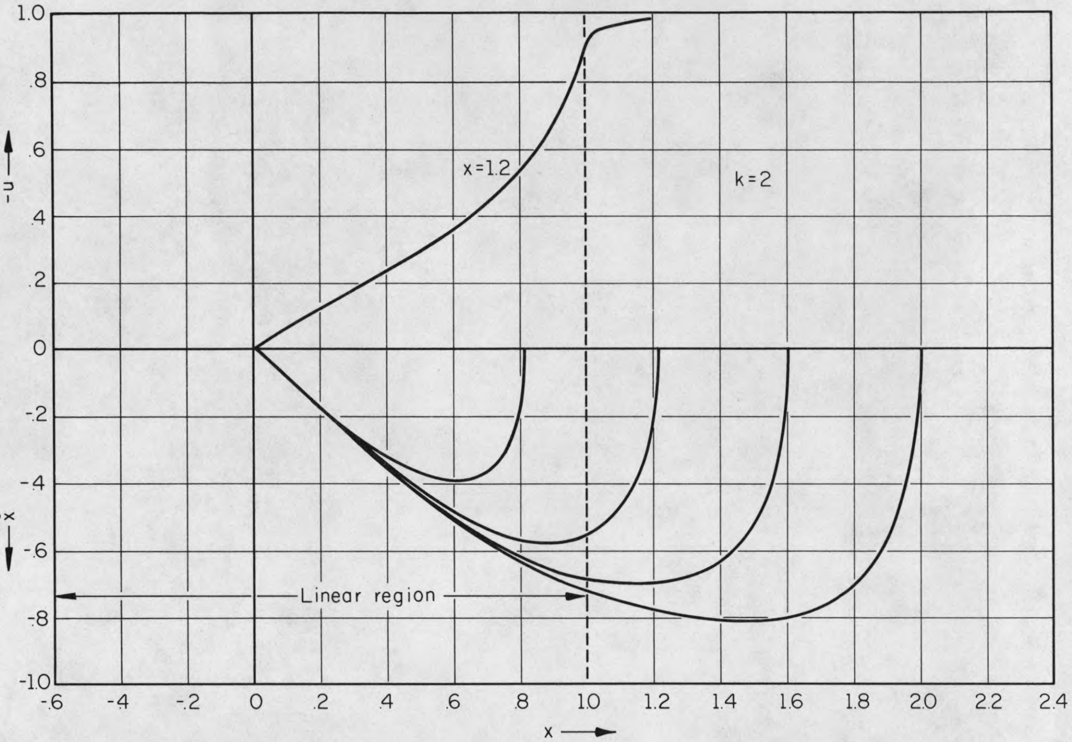

Fig. 3. x_2 and u verses x_1 for the second-order example with various values of gain: (a) k = 2.

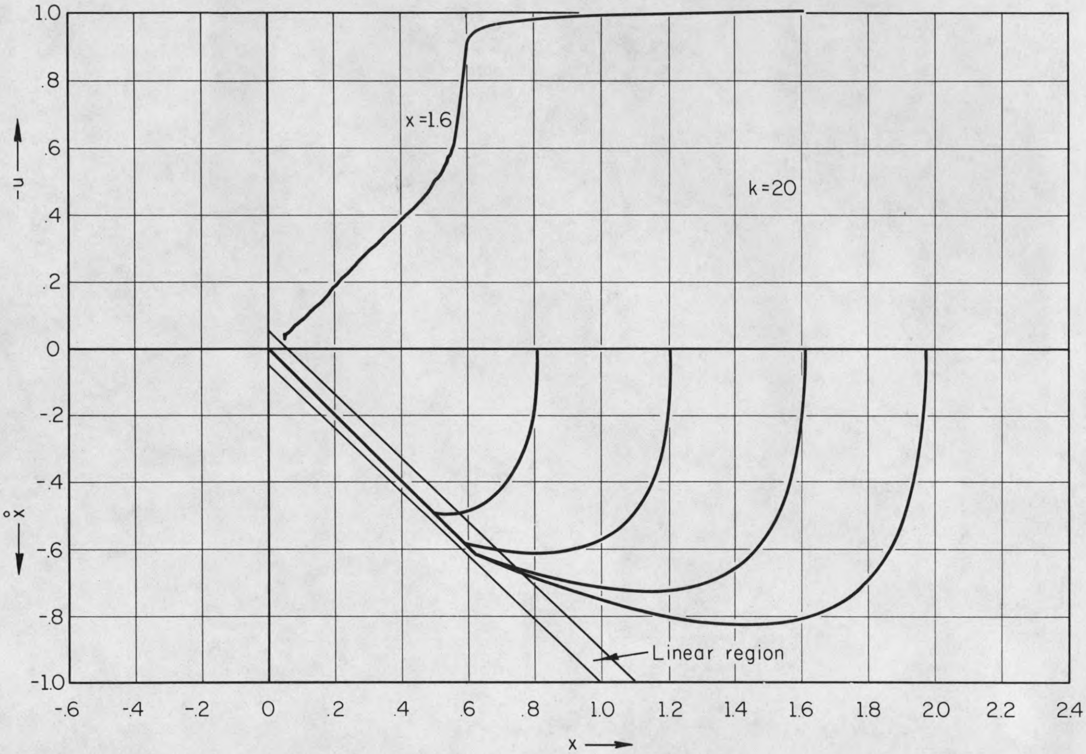

Fig. 3. x_2 and u verses x_1 for the second-order example with various values of gain: (b) k = 20.

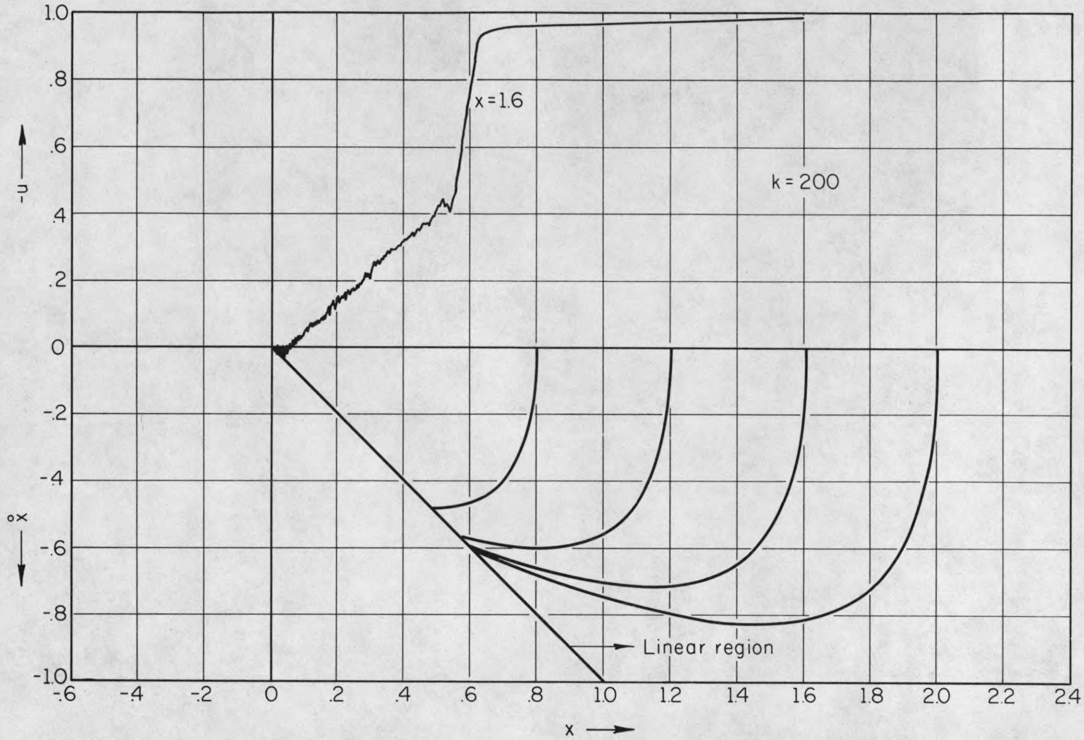

Fig. 3. x_2 and u verses x_1 for the second-order example with various values of gain: (c) k = 200.

IV. EXTENSION--NONLINEAR, TIME-VARYING SYSTEMS

The extension of the above results to nonlinear, time-varying systems is quite easily undertaken. Under the assumption that the performance index is the same time-invariant quadratic form, we obtain the same stable Euler equation and the same simple expression for $\underline{\Psi}$ as above. It is only in the implementation of the feedback control that the newly hypothesized qualities of the system enter the picture.

Suppose, for example, that we are provided with a single-input system characterized by an n^{th} order nonlinear, time-varying differential equation:

$$x^{(n)} = - \sum_{i=i}^{n} a_i (\underline{x},t)\, x^{(i-1)} + u, \tag{29}$$

where

$$\underline{x} = \begin{bmatrix} x \\ x^{(1)} \\ x^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ x^{(n-1)} \end{bmatrix} ; \tag{30}$$

Moreover, the control u is constrained, as in (2) - (4), and the performance criterion demands the minimization of the functional (5). The Euler equation and auxiliary vector $\underline{\Psi}$ are exactly as found in Appendix B; the optimal control is given by

$$\underline{c} = \underline{a}(\underline{x},t) - \hat{\underline{h}} - k\underline{h}, \tag{31}$$

where

$$\underline{a}(\underline{x},t) = \begin{bmatrix} a_1(\underline{x},t) \\ a_2(\underline{x},t) \\ \cdot \\ \cdot \\ \cdot \\ a_n(\underline{x},t) \end{bmatrix} . \qquad (32)$$

As before, the optimal is achieved by making the gain (k) as large as possible (or desirable). The significant factor here is that the portion of the control ($\underline{c}_{ns}^T \underline{x} = k\underline{h}^T\underline{x}$) which drives the system to the singular hyperplane (and, as such, defines the switching curve) is linear and time-invariant. Moreover, this portion of the control is solely dependent on the performance index; consequently, the closed-loop system should be relatively insensitive to plant parameter variations for large gain (k).

V. CONCLUSION

The advantages of the optimal control scheme presented here are manifest, to name but a few:

1. Ease of implementation;
2. Indifference to initial conditions, i.e., the feedback control is the same regardless of starting point;
3. Applicability to optimal control of nonlinear, time-varying systems.

This form of control should enjoy wide application as it is the theoretical optimum tempered with practical constraints.

ACKNOWLEDGMENT

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APPENDIX A. SUB-OPTIMAL LINEAR SWITCHING

Consider the following problem.

Given the system equation

$$\dot{\underline{x}} = A\underline{x} + \underline{b}\,u, \tag{A.1}$$

find $\underline{u} \in U$ (U is defined as the set of all piecewise continuous functions u(t) such that

$$|u(t)| \leq 1 \;) \tag{A.2}$$

such that the functional

$$J(u) = \int_0^\infty \underline{x}^T Q\, \underline{x}\, dt \tag{A.3}$$

is minimized. Under the assumption of complete controllability for the system (A.1), we can take without loss of generality [5]

$$A = \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & 0 \\ . & & & & & & \\ . & & & & & & \\ . & & & & & & \\ -a_1 & -a_2 & -a_3 & . & . & . & -a_n \end{bmatrix}, \tag{A.4a}$$

$$\underline{b} = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{bmatrix}, \tag{A.4b}$$

and

$$\underline{x} = \begin{bmatrix} x \\ \dot{x} \\ \cdot \\ \cdot \\ \cdot \\ x^{(n-1)} \end{bmatrix}. \tag{A.4c}$$

We assume that A(the uncontrolled system) is stable and that the matrix Q (A.3) is positive definite; therefore, the equation

$$PA + A^T P = -2Q \tag{A.5}$$

has a unique positive definite solution, P [9]. Let

$$V(\underline{x}) = \underline{x}^T P\underline{x} ; \tag{A.6}$$

then, along the solution of (1)

$$\frac{dV(\underline{x})}{dt} = 2\underline{x}^T P\dot{\underline{x}} = 2\underline{x}^T PA\underline{x} + 2\underline{x}^T P\underline{b}u. \tag{A.7}$$

Moreover, because of (A.5)

$$\frac{dV(\underline{x})}{dt} = -2\underline{x}^T Q\underline{x} + 2\underline{x}^T P\underline{b}\, u, \tag{A.8}$$

which upon integration (0 to ∞) yields

$$\int_o^\infty \underline{x}^+ Q\underline{x}\, dt = -\frac{1}{2}\int_o^\infty \frac{dV}{dt}\, dt + \int_o^\infty \underline{x}^T P\underline{b}u\, dt. \tag{A.9}$$

Hence

$$J(u) = \frac{1}{2}\, \underline{x}_o^T P\underline{x}_o + S(u) \tag{A.10}$$

(here we are assuming the controlled system to be stable), where

$$S(u) = \int_{0}^{\infty} \underline{x}^T P \underline{b} u \, dt. \tag{A.11}$$

Clearly, the control u which minimizes J(u) is the same control which minimizes S(u).

By the Maximum Principle of Pontryagin [7], the optimal control u must maximize the Hamiltonian

$$\mathcal{H}(\underline{\Psi}, \underline{x}, u) = \underline{\Psi}^T A \underline{x} + \underline{\Psi}^T \underline{b}u - x^T P \underline{b}u. \tag{A.12}$$

Hence, when the system trajectory is not on a singular surface (defined by $\underline{b}^T(\underline{\Psi} - P\underline{x}) = 0$), the optimal control is given by

$$u = \text{sign}\, [\underline{b}^T(\underline{\Psi} - P\underline{x})] . \tag{A.13}$$

From the definition of $\underline{\Psi}$,

$$\frac{d\underline{\Psi}}{dt} = -A^T \underline{\Psi} + P \underline{b} u; \tag{A.14}$$

hence,

$$\underline{\Psi}(t) = e^{-A^T t} \underline{\Psi}_o + \int_{o}^{t} e^{-A^T(t-\tau)} P\underline{b} \, u(\tau) \, d\tau. \tag{A.15}$$

Jen Wei's [1] solution to the optimization problem is simply to take

$$u = - \text{sign}\, \underline{b}^T P \underline{x}, \tag{A.16}$$

which would be correct if

$$\underline{\Psi}(t) = \underline{0}. \tag{A.17}$$

The control (A.16) is also Bass' [4] suggestion for a simple control scheme. The only difference is that Jen Wei considers the trajectory which keeps $\underline{b}^T P \underline{x} = 0$ to be optimal as well (he called it the slippage mode); he then finds the associated linear control--on the other hand, Bass simply lets $u = 0$ whenever $\underline{b}^T P \underline{x} = 0$.

APPENDIX B. SINGULAR SURFACES

Without loss of generality we can put our assumed completely controllable system (1) into companion form [5]:

$$\underline{\dot{x}} = \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 1 \\ -a_1 & -a_2 & -a_3 & . & . & . & -a_n \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{bmatrix} u; \tag{B.1}$$

we will assume that this is the case. It is obvious now that the maintenance of a singular solution over a finite time interval requires that the n^{th} component of $\underline{\Psi}$ and its derivatives be zero:

$$\Psi_n = \Psi_n^{(1)} = \Psi_n^{(2)} = . . . = 0 \tag{B.2}$$

From (7), the expression for the first derivative of the i^{th} component of $\underline{\Psi}$ is

$$-\dot{\Psi}_i = \frac{\partial \mathcal{H}}{\partial x_i} , \tag{B.3}$$

and the i^{th} derivative is

$$\frac{d^i}{dt^i} (-\Psi_i) = \frac{d^{i-1}}{dt^{i-1}} \frac{\partial \mathcal{H}}{\partial x_i} . \tag{B.4}$$

Consequently,

$$\sum_{i=1}^{n} (-1)^i \frac{d^i(-\Psi_1)}{dt^i} = \sum_{i=1}^{n} (-1)^i \frac{d^{i-1}}{dt^{i-1}} \frac{\partial \mathcal{H}}{\partial x_i} , \tag{B.5}$$

or, in view of (6),

$$\sum_{i=1}^{n} (-1)^i \frac{d^{i=1}}{dt^i} \frac{\partial(\underline{x}^T Q\underline{x})}{\partial x_i} = \sum_{i=1}^{n} (-1)^i \frac{d^{i-1}}{dt^{i-1}} \left[\frac{\partial}{\partial x_i} (\underline{\Psi}^T A\underline{x} + \underline{\Psi}^T \underline{b}f(v) + \frac{d\Psi_i}{dt} \right] \quad (B.6)$$

The singularity requirement (B.2) reduces (B.6) to

$$\sum_{i=1}^{n} (-1)^i \frac{d^{i-1}}{dt^{i-1}} \frac{\partial(\underline{x}^T Q\underline{x})}{\partial x_i} = \sum_{i=1}^{n} (-1)^i \frac{d^{i-1}}{dt^{i-1}} \left[\frac{\partial}{\partial x_i} (\underline{\Psi}^T A\underline{x}) + \frac{d\underline{\Psi}_i}{dt} \right] \quad (B.7)$$

Moreover, because the system is in companion form (B.1), we find that

$$\frac{\partial}{\partial x_i} (\underline{\Psi}^T A\underline{x}) = \frac{\partial}{\partial x^{(i-1)}} (\underline{\Psi}^T A\underline{x}) = \Psi_n a_i + \Psi_{i-1} \, . \quad (B.8)$$

This result, when substituted into (B.7), yields

$$\sum_{i=1}^{n} (-1)^i \frac{d^{i-1}}{dt^{i-1}} \frac{\partial(\underline{x}^T Q\underline{x})}{\partial x_i} = \sum_{i=1}^{n} (-1)^i a_i \Psi_n^{(i-1)} \, . \quad (B.9)$$

But, from (B.2), the sum on the right hand side is zero; consequently, the singular surfaces are given by the solutions of

$$\sum_{i=1}^{n} (-1)^i \frac{d^{i-1}}{dt^{i-1}} \frac{\partial(\underline{x}\ Q\underline{x})}{\partial x_i} = 0 \quad (B.10a)$$

or

$$\sum_{i=1}^{n} (-1)^i \frac{d^{i-1}}{dt^{i-1}} \frac{\partial(\underline{x}\ Q\underline{x})}{\partial x^{(i-1)}} = 0 \, , \quad (B.10b)$$

which is the Euler equation and the fundamental necessary condition for an extremum of (5). We will not discuss sufficient conditions--in general they are more easily determined for a specific problem than for an overall class of problems.

It has been shown [10] that the roots of the characteristic polynomial associated with the Euler equation (B.10) occurs in quadrantal symmetry in the complex plane (this fact follows simply from the self-adjoint character of the Euler equation); consequently, we need only factor out the left half plane roots to obtain the stable singular solutions. The stable singular solutions satisfy the linear differential equation of order (n-1)

$$\sum_{i=1}^{n} h_i \, x^{(i-1)}(t) = 0, \tag{B.11}$$

which describes an (n-1)-dimensional hyperplane (the stable singular surface) in the n-dimensional state space.

FOOTNOTES

1. For a short derivation and discussion of the results of Jen-Wei, see Appendix A.

2. The functional $J(\underline{c})$ is indicated as a function of the vector $\underline{c}$ because the c_i's are the only control variables, which follows from the problem formulation, (1) - (4).

3. The system is assumed completely controllable; hence the state equations can always be transformed to companion form (see [5] and Appendix B).

4. $\underline{h}^T \underline{x} = 0$ implies $\underline{u} = (\underline{a} - \hat{\underline{h}})^T \underline{x}$, which is the case on the singular surface.

5. The alternate solution, $c_1 = +\infty$, renders the system unstable.

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